CHAPTER3

COMPLEX INTEGRATION

We defined the differentiation with respect to z in the case of a complex function, since z is a function of two independent variables x and y, i.e. $z = x + jy$. Complex integration is approached in the same way.

$$z = x + jy \text{ and } w = f(z) = u + jv$$

where u and v are also functions of x and y. Also

$$dz = dx + j\,dy \quad \text{and} \quad dw + du + j\,dv$$

$$\int w\,dz = \int f(z)\,dz = \int (u + jv)(dx + jdy)$$
$$= \int \{(u\,dx - v\,dy) + j(v\,dx + u\,dy)\}$$

Then

$$\int f(z)\,dz = \int (u\,dx - v\,dy) + j(v\,dx + u\,dy)$$

That is, the integral reduces to two real-variable integrals

$$\int (u\,dx - v\,dy) \quad \text{and} \quad \int (v\,dx + u\,dy)$$

Note that each of these two integrals is of the general form

CamScanner

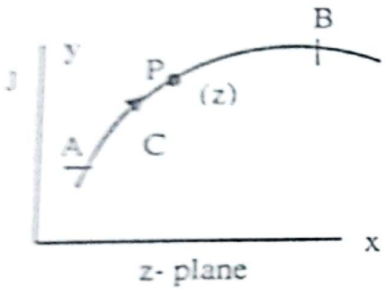
$$\int (P\,dx + Q\,dy)$$

which we met before during our work on line integrals and complex plane, this rather neatly leads us into contour integ

1 ■CONTOUR INTEGRATION►

1. LINE INTEGRALS IN THE Z-PLANE►

If z moves along the curve c in the z-plane and at each po has associated with it a function of z, i.e. $f(z)$, then sum



$f(z)$ for all such points be and B means that we are eval line integral in the z-plane A$(z = z_1)$ and B$(z = z_2)$ a curve c, i.e. we are ev $\int_C f(z)\,dz$ where c is the pa path joining A to B.

The evaluation of line integrals in the complex plane is kn contour integration. Let us see how it works in practice.

EXAMPLE 1►

Evaluate the integral $\int_C f(z)\,dz$ where $f(z) = (z - j)^2$ and the straight line joining A$(z = 0)$ to B$(z = 1 + j2)$.

Solution

$$z = x + \mathrm{j}\,y; \quad dz = dx + \mathrm{j}\,dy$$

$$f(z) = (z-\mathrm{j})^2 = \{x + \mathrm{j}(y-1)\}^2 = x^2 - (y-1)^2 + \mathrm{j}2x(y-1)$$

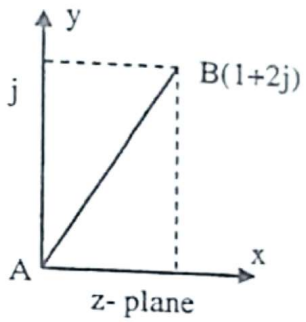
$$I = \int \{(x^2 - y^2 + 2y - 1) + \mathrm{j}(2xy - 2x)\}\{dx + \mathrm{j}\,dy\}$$

$$I = \int \{(x^2 - y^2 + 2y - 1)\,dx - (2xy - 2x)\,dy\}$$
$$+ \mathrm{j}\int \{(2xy - 2x)\,dx + (x^2 - y^2 + 2y - 1)\,dy\}$$

Now the equation of AB is

$y = 2x \Rightarrow dy = 2\,dx$

and substituting these in the expression for I, between the limits $x = 0$ and $x = 1$ gives



$$I = \int_0^1 \{(x^2 - 4x^2 + 4x - 1)\,dx - (4x^2 - 2x)\,2\,dx\}$$
$$+ \mathrm{j}\int_0^1 \{(4x^2 - 2x)\,dx + (2x^2 - 8x^2 + 8x - 2)\,dx\}$$

$$I = \int_0^1 (11x^2 + 8x - 1)\,dx + \mathrm{j}\int_0^1 (-2x^2 + 6x - 2)\,dx$$

and this, by elementary integration, gives

$$I = \tfrac{1}{3}(-2 + \mathrm{j})\ \#$$

CamScanner

Remember that, in general, the value of a line integral depends on the path of integration between the end points, but that the integral $\int (P\,dx + Q\,dy)$ is independent of the path of integration in a simply connected region $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ throughout the region.

In our example

$$I = \int \left\{ \left(x^2 - y^2 + 2y - 1\right) dx - (2xy - 2x)\, dy \right\} + \mathrm{j} \int \left\{ (2xy - 2x)\, dx + \left(x^2 - y^2 + 2y - 1\right) dy \right\}$$

$$\equiv I_1 + \mathrm{j}\, I_2$$

If we apply the test to I_1 we get

$$I_1 = \int \left\{ \left(x^2 - y^2 + 2y - 1\right) dx - (2xy - 2x)\, dx \right\}$$

$$\equiv \int (P\,dx + Q\,dy)$$

$$\left.\begin{array}{ll} P = x^2 - y^2 + 2y - 1 & \therefore \dfrac{\partial P}{\partial y} = -2y + 2 \\ Q = -2xy + 2x & \therefore \dfrac{\partial Q}{\partial x} = -2y + 2 \end{array}\right\} \therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Similarly

$$I_2 = \int \left\{ (2xy - 2x)\, dx + \left(x^2 - y^2 + 2y - 1\right) dx \right\}$$

$$\equiv \int (P\,dx + Q\,dy)$$

$$\left.\begin{array}{ll} P = 2xy + 2x & \therefore \dfrac{\partial P}{\partial y} = 2x \\ Q = x^2 - y^2 + 2y - 1 & \therefore \dfrac{\partial Q}{\partial x} = 2x \end{array}\right\} \therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

CamScanner

Therefore, in this example, the value of the line integral is independent of the path of integration.
Just to satisfy our conscience, determine the value of the line integral between the same two end points, but along the parabola $y = 2x^2$

$f(z) = (z - j)^2$

$y = 2x^2 \quad \therefore dy = 4x\,dx$

As before we have

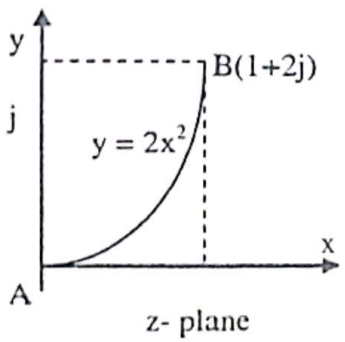
$$I = \int \{(x^2 - y^2 + 2y - 1)dx - (2xy - 2x)dy\}$$
$$+ j\int \{(2xy - 2x)dx + (x^2 - y^2 + 2y - 1)dy\}$$

Substituting $y = 2x^2$ and $dy = 4x\,dx$, the evaluation gives

$$I = \int_0^1 \{(x^2 - 4x^4 + 4x^2 - 1)dx - (4x^3 - 2x)4x\,dx\}$$
$$+ j\int_0^1 \{(4x^3 - 2x)dx + (x^2 - 4x^4 + 4x^2 - 1)4x\,dx\}$$
$$= \int_0^1 (-20x^4 + 13x^2 - 1)\,dx + j\int_0^1 (-16x^5 + 24x^3 - 6x)dx$$

$$I = \tfrac{1}{3}(-2 + j)$$

which is, of course, the same result as before.



z- plane

CamScanner

EXAMPLE 2▶

Show that $\int_0^1 (t-j)^3\,dt = -\frac{5}{4}$.

SOLUTION.

We write the integrand in terms of its real and imaginary parts, i.e.,

$$f(t) = (t-j)^3 = t^3 - 3t + j(-3t^2+1)$$

Here, $u(t) = t^3 - 3t$ and $v(t) = -3t^2 + 1$.

The integrals of u(t) and v(t) are

$$\int_0^1 u(t)\,dt = \int_0^1 (t^3 - 3t)\,dt = \left[\frac{t^4}{4} - \frac{3t^2}{2}\right]_0^1 = -\frac{5}{4}, \text{ and}$$

$$\int_0^1 v(t)\,dt = \int_0^1 (-3t^2+1)\,dt = \left[-t^3 + t\right]_0^1 = 0$$

Hence,

$$\int_0^1 (t-j)^3\,dt = \int_0^1 u(t)\,dt + j\int_0^1 v(t)\,dt = -\frac{5}{4}$$

EXAMPLE 3▶

Show that $\int_0^{\pi/2} e^{t+jt}\,dt = \frac{1}{2}\left(e^{\pi/2} - 1\right) + \frac{j}{2}\left(e^{\pi/2} + 1\right)$

SOLUTION

$$\int_0^{\pi/2} e^{t+jt}\,dt = \int_0^{\pi/2} e^t\, e^{jt}\,dt = \int_0^{\pi/2} e^t\,[\cos t + j\sin t]\,dt$$

CamScanner

$$= \int_0^{\pi/2} e^t \cos t \, dt + j \int_0^{\pi/2} e^t \sin t \, dt$$

We can evaluate each of the integrals via integration by parts. We obtain:

$$\int_0^{\pi/2} e^t \cos t \, dt = \frac{1}{2}\left(e^{\pi/2} - 1\right) ,$$

and

$$\int_0^{\pi/2} e^t \sin t \, dt = \frac{1}{2}\left(e^{\pi/2} + 1\right)$$

Therefore,

$$\int_0^{\pi/2} e^{t+jt} \, dt = \frac{1}{2}\left(e^{\pi/2} - 1\right) + \frac{j}{2}\left(e^{\pi/2} + 1\right) \blacksquare$$

2 ■CAUCHY'S THEOREM►

We have already seen that if $w = f(z)$ where, $w = u + \mathrm{j}v$ and

$$z = x + \mathrm{j}y \Rightarrow dz = dx + \mathrm{j}\, dy$$

and

$$\int f(z)dz = \int (u + \mathrm{j}v)(dx + \mathrm{j}\, dy)$$

$$= \int (u\, dx - v dy) + \mathrm{j} \int (v dx + u\, dy)$$

If c is a simply connected closed curve as the path of integration, then

$$\oint_c f(z)dz = \oint_c (u\, dx - v dy) + \mathrm{j} \oint_c (v dx + u\, dy)$$

CamScanner

Applying Green's theorem to each of the two integrals on the r hand side in turn, we have:

(a) $\oint_c (u\,dx - v\,dy) = \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy$

where S is the region enclosed by the curve c.

Also, if $f(z)$ is regular at every point within and on c, then

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ and therefore } \quad -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = 0$$

$$\therefore \oint_c (u\,dx - v\,dy) = 0 \qquad (1)$$

(b) Similarly, with the second integral, we have

$$\oint_c (v\,dx - u\,dy) = 0$$

since $\oint_c (v\,dx - u\,dy) = \iint_S \left(-\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy$

Again, if $f(z)$ is regular at every point within and on c, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and therefore } \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

$$\therefore \oint_c (v\,dx + u\,dy) = 0 \qquad (2)$$

Combining the two results (1) and (2) we have the following rem:

Cauchy's theorem ▶ If $f(z)$ is regular at every point within on a simply connected closed curve c, then

$$\oint_c f(z)\,dz = 0 \qquad (3)$$

CamScanner

EXAMPLE 1► Verify Cauchy's theorem by evaluating the integral $\oint_c f(z)\,dz$,where $f(z)=z^2$ around the square formed by joining the points : $z=1,\ z=2,\ z=2+\mathrm{j},\ z=1+\mathrm{j}$.

SOLUTION

$$z^2 = x^2 - y^2 + \mathrm{j}2xy$$

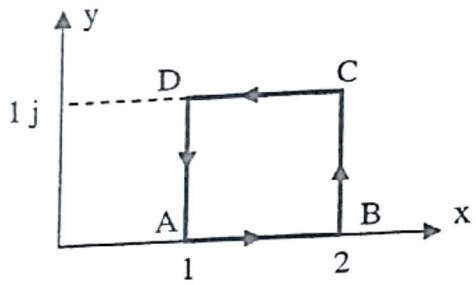
$$z = x + \mathrm{j}y \quad \Rightarrow \quad dz = dx + \mathrm{j}\,dy$$

$$\oint_c f(z)\,dz = \oint_c z^2\,dz = \oint_c \{x^2 - y^2 + \mathrm{j}2xy\}\{dx + \mathrm{j}\,dy\}$$

$$= \oint_c \{(x^2 - y^2)\,dx - 2xy\,dy\}$$

$$+ j\oint_c \{2xy\,dx + (x^2 - y^2)\,dy\}$$

We now take each of the sides in turn.



(a) AB: $y=0 \qquad \therefore dy=0$

$$\therefore \int_{AB} f(z)\,dz = \int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

(b) BC: $x=2 \qquad \therefore dx=0$

$$\therefore \int_{BC} f(z)\,dz = \int_0^1 (-4y\,dy) + \mathrm{j}\int_0^1 (4 - y^2)\,dy$$

CamScanner

$$= \left[-2y^2\right]_0^1 + j\left[4y - \frac{y^3}{3}\right]_0^1$$

$$= -2 + j\left(4 - \frac{1}{3}\right) = -2 + j\frac{11}{3}$$

Continuing in the same way, the results for the remaining t sides are

(c) CD: $y = 1 \qquad \therefore dy = 0$

CD: $-\frac{4}{3} - j3$, DA: $1 - j\frac{2}{3}$

$$\therefore \int_{CD} f(z)\,dz = \int_1^2 (x^2 - 1)\,dx + j\int_2^1 2x\,dx$$

$$= \left[\frac{x^3}{3} - x\right]_2^1 + j\left[x^2\right]_2^1 = -\frac{4}{3} - 3j$$

(d) DA $x = 1 \qquad \therefore dx = 0$

$$\therefore \int_{DA} f(z)\,dz = \int_1^0 (-2y\,dy) + j\int_1^0 (1 - y^2)\,dy$$

$$= \left[-y^2\right]_1^0 + j\left[y - \frac{y^3}{3}\right]_1^0 = 1 - j\frac{2}{3}$$

So, collecting the four results, we obtain:

$$\oint_C f(z)\,dz = \frac{7}{3} + \left(-2 + j\frac{11}{3}\right) + \left(-\frac{4}{3} - j3\right) + \left(1 + j\frac{2}{3}\right) = 0.$$

$$\oint_C f(z)\,dz = 0$$

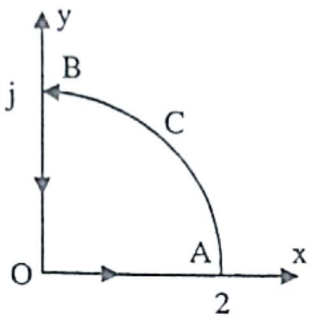
CamScanner

EXAMPLE 2 ▶ A region in the z-plane has a boundary c consisting of :

(a) OA joining $z = 0$ to $z = 2$

(b) AB a quadrant of the circle $|z| = 2$ from $z = 2$ to $z = j2$

(c) BO joining $z = j2$ to $z = 0$.



Verify Cauchy's theorem by evaluating the integral $\int_c (z^2+1)dz$

(i) along the arc from A to B, (ii) along BO and OA.

SOLUTION

$$f(z) = z^2 + 1 = (x + jy)^2 + 1 = (x^2 - y^2 + 1) + j2xy$$

$$z = x + jy \quad \Rightarrow \quad dz = dx + j\,dy$$

So the general expression for $\int f(z)dz.$ is:

$$\int f(z)\,dz = \int \{(x^2 - y^2 + 1) + j2xy\}\{dx + j\,dy\}$$

(i) Arc AB:

$$x^2 + y^2 = 4 \Rightarrow \quad y = \sqrt{4 - x^2} \Rightarrow dy = \frac{-x}{\sqrt{4 - x^2}}dx$$

$$\therefore \int_{AB} f(z)dz = \int_2^0 \left\{ (x^2 - 4 + x^2 + 1)dx - 2x\sqrt{4 - x^2}\left(\frac{(-x)}{\sqrt{4 - x^2}}\right)dx \right\}$$

$$+ \int_2^0 \left\{ 2x\sqrt{4 - x^2}\,dx + (x^2 - 4 + x^2 + 1)\left(\frac{(-x)}{\sqrt{4 - x^2}}\right)dx \right\}$$

$$= \int_2^0 (4x^2 - 3)dx + j\int_2^0 \frac{11x - 4x^3}{\sqrt{4 - x^2}}dx = -\frac{14}{3} + jI_1$$

CamScanner

Now we must attend to

$$I_1 = \int_2^0 \frac{11x - 4x^3}{\sqrt{4-x^2}}\,dx .$$

Substituting $x = 2\sin\theta$ and $dx = 2\cos\theta\, d\theta$ with appropriate limits we have

$$I_1 = \int_{\pi/2}^{0} \left(\frac{22\sin\theta - 32\sin^3\theta}{2\cos\theta} \right) 2\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \left(32\sin^3\theta - 22\sin\theta\right) d\theta$$

$$= 32\frac{2}{(3)(1)} + \left[22\cos\theta\right]_0^{\pi/2} = \frac{64}{3} - 22 = -\frac{2}{3}$$

Thus:

$$\int_{AB} f(z)\,dz = -4\frac{2}{3} - \mathrm{j}\frac{2}{3} = -\frac{2}{3}(7+\mathrm{j})$$

(ii) Along BO and OA.

BO: $x = 0 \qquad \therefore dx = 0$

$$\therefore \int_{BO} f(z)\,dz = \mathrm{j}\int_2^0 \left(1 - y^2\right) dy = \mathrm{j}\left[y - \frac{y^3}{3} \right]_2^0 = \mathrm{j}\frac{2}{3}$$

OA: $y = 0 \qquad \therefore dy = 0$

$$\therefore \int_{OA} f(z)\,dz = \mathrm{j}\int_0^2 \left(x^2 + 1\right) dx = \left[\frac{x^3}{3} + x \right]_0^2 = 4\frac{2}{3}$$

Collecting the results together, therefore

$$\int_{AB} f(z)\,dz = -\frac{14}{3} - \mathrm{j}\frac{2}{3}$$

CamScanner